Tutorial: Law of Large Numbers and Central Limit Theorem

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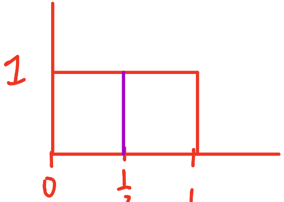
$Pr(X = \frac{1}{2}) = 0$

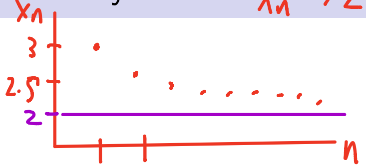


$X \sim U[0,1]$

Convergence in Probability

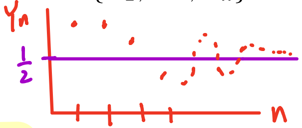
$X_n \to 2$ as $n \to \infty$

① $\underbrace{X_n = 2 + \frac{1}{n}}_{\text{deterministic}}$



- Consider a sequence of n random variables $\{X_1, \ldots, X_n\}$ and some constant c

② $Y_n = \dfrac{\text{\# of tails in n flips}}{n}$



- The sequence converges in probability to c if $Pr(X_n \to c) = 1$ as $n \to \infty$

① $X_n \to 2 \Rightarrow Pr(X_n \to 2) = 1 \Rightarrow X_n \xrightarrow{P} 2$

② $Pr(Y_n \to \frac{1}{2}) = 1 \Rightarrow Y_n \xrightarrow{P} \frac{1}{2}$

Law of Large Numbers

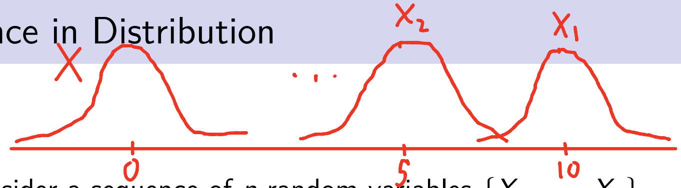
$Pr(\bar{X} \to \mu) = 1$

- Suppose $\bar{X}$ is estimator for μ
- Law of large numbers: as $n \to \infty$ then $\bar{X} \xrightarrow{p} \mu$ if
  - Sample is IID from the population
  - $E(X_i) = \mu < \infty$
- What property of the mean estimator does the LLN imply?

↳ $\bar{X}$ is consistent since LLN $\Rightarrow \bar{X} \xrightarrow{p} \mu$

↳ $E(\bar{X}) = \mu$, $V(\bar{X}) = \frac{\sigma^2}{n} \underset{n \to \infty}{\longrightarrow} 0$

Convergence in Distribution



- Consider a sequence of n random variables $\{X_1, \ldots, X_n\}$, where each $X_i \sim F_i$ and there is some RV $X \sim F$

  $\hookrightarrow X_n = N(0,1) + \frac{10}{n} \xrightarrow[n \to \infty]{d} N(0,1) = X$

- The sequence of RVs convergence in distribution to X if $F_n(x) \to F(x)$ as $n \to \infty$

- What can you say about the t distribution with $df = n$?

  $\underbrace{t_n}_{X_n} \xrightarrow{d} \underbrace{N(0,1)}_{X}$

Central Limit Theorem

- Central limit theorem says that $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ under:
  - The sample is independently and identically drawn (IID) from the population
  - Sample size is sufficiently large
- How does the CLT relate to convergence in distribution?

$\bar{X} \sim N(\mu, \frac{\sigma^2}{n}) \Rightarrow \bar{X} - \mu \sim N(0, \sigma^2/n)$

$\Rightarrow \underbrace{\sqrt{n}(\bar{X} - \mu)} \sim N(0, \sigma^2)$, CLT $\Rightarrow Z_n \xrightarrow{d} N(0, \sigma^2)$

Asymptotic and Estimator Properties

. $\hat{X}_n$ is estimator for θ

$Pr(\hat{X}_n \to \theta) = 1$

- Consistency: $\widehat{X_n} \xrightarrow{p} \theta$ as $n \to \infty$

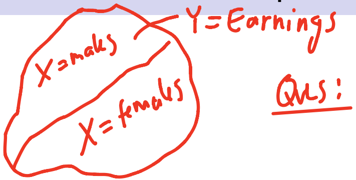
<u>Ex.</u> $\overline{X} \xrightarrow{p} \mu$, $\hat{p} \xrightarrow{p} p$, $\hat{s}^2 \xrightarrow{p} \sigma^2$, $\underbrace{\hat{\beta} \xrightarrow{p} \beta}_{X \perp \varepsilon}$

- Asymptotic Sampling distribution: Distribution of $\widehat{X_n}$ in large samples

↳ Used for Hypo. testing to determine dbn of $\hat{X}_n$ under H_0

↳ $\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$, $\hat{p} \sim N\left(p, \frac{p(1-p)}{n}\right)$, $\hat{\beta} \sim N\left(\beta, \frac{s^2}{\sum_i (X_i - \overline{X})^2}\right)$

Law of Iterated Expectations



Ques: Using $\underset{(n_{males})}{\bar{Y}_{males}}$, $\underset{(n_{females})}{\bar{Y}_{females}} \Rightarrow \bar{Y}$?

- LIE Property: $E(E(Y|X)) = E(Y)$

$$\underbrace{\bar{Y}}_{E(Y)} = \underbrace{\frac{n_{males}}{n_{total}}\underbrace{\bar{Y}_{males}}_{E(Y|X=m)} + \frac{n_{females}}{n_{total}}\underbrace{\bar{Y}_{females}}_{E(Y|X=f)}}_{E(E(Y|X))}$$

- Show $E(u|x) = 0$ implies $E(ux) = 0$

$$E(ux) \underset{\text{LIE}}{=} E(E(ux|x)) = E(x\underbrace{E(u|x)}_{0})$$

$$= E(x \cdot 0) = 0$$